

## A RECONSTRUCTION METHOD OF GENERALIZED SAMPLING BASED ON GENERALIZED INVERSE

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### Abstract

This paper considers the problem of reconstructing a class of generalized sampled signals of which a special case occurs in, *e.g.*, a generalized sampling system due to non-ideal analysis basis functions. To this end, we propose an improved reconstruction system and a reconstruction algorithm based on generalized inverse, which can be viewed as a reconstruction method that reduces reconstruction error as well. The key idea is to add an additional channel into a generalized sampling system and apply the generalized inverse theory to the reconstruction algorithm. Finally, the approach is applied, respectively, to an oscilloscope, which shows the proposed method yields better performance as compared to the existing technique.

Keywords: Hilbert space, generalized sampling, reconstruction, generalized inverse.

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### 1. Introduction

The generalized sampling problem is to reconstruct an unknown continuously defined input function  $x(t) \in W$  from the samples of the responses of  $L$  linear shift-invariant systems sampled at  $1/L$ -th the reconstruction rate. Papoulis provided an elegant solution for the particular case where the space  $W$  is a class of band-limited functions [1]. Recently, we have extended the scope of Papoulis' theory by introducing a formal distinction between the input space and the reconstruction space. Then, the generalized sampling can then be represented as the inner products of the input signal with a set of sampling vectors, which span the sampling space  $S$  [2–11]. Examples include multi-resolution [10] spline decompositions [4]. And, reconstruction is obtained by forming linear combinations of a set of reconstruction vectors that span a space  $W$ . The reconstruction was obtained by first processing the samples by a digital correction filter [12], then forming linear combinations of a set of reconstruction vectors that span a space  $W$ . The traditional reconstruction method schemes based on the analysis basis functions are ideal.

In this paper, we design reconstruction strategies for the generalized sampling scheme, where we treat the problem of reconstructing a class of generalized sampled signals of which a special case occur in, *e.g.*, generalized sampling system dues to non-ideal analysis basis functions. To ensure that the reconstruction  $\hat{x}(t)$  is close to  $x(t)$ , we may try to add an additional channel into the generalized sampling system and use a generalized inverse theory.

The paper is organized as follows. The general sampling we treat in the paper is introduced in Section 2. Section 3 shows that the reconstruction method is obtained by generalized inverse theory. In Section 4 we analyze the reconstruction error resulting from the proposed approach, and compare it with the error from the traditional method. Experiment results are presented in Section 5.

## 2. Generalized sampling

In this section, we briefly review the generalized sampling theory that was developed in [1]. The corresponding system is schematically represented in Fig. 1. The continuous-time signal is convolved with a bank of analysis filters  $h_i(t) = s_i(-t)$   $i = 1, 2, \dots, L$ , the responses of which are sampled at  $1/L$ -th the reconstruction rate to yield the measurement vector  $d(n) = (d_1(n) \ d_2(n) \ \dots \ d_L(n))^T$ . Our main reason for introducing the time-reversed analysis functions is that we can describe the measurement process in terms of inner products.

$$d_k[n] = Sx = \langle x(t), s_k(t - nT) \rangle \quad k = 1, 2, \dots, L. \quad (1)$$

Here, we will define signal models by considering the representation space:

$$W = \left\{ \sum_{i=1}^L \sum_{n \in \mathbb{Z}} c_i[n] \varphi_i(t - nT) : c_i[n] \in l^2 \right\}, \quad (2)$$

where  $\{\varphi_i(t)\}_{i=1,2,\dots,L}$  is a given generating function sequence. Our only restriction on the choice of the generating function sequence  $\{\varphi_i(t)\}_{i=1,2,\dots,L}$  is that  $W$  is a closed subspace of  $l^2$  with as its orthonormal Riesz basis. In other words, there must exist two constants  $A$  and  $B$  ( $0 < A \leq B < \infty$ ), such that:

$$A \sum_{i=1}^L \sum_{n \in \mathbb{Z}} |c_i[n]|^2 \leq \left\| \sum_{i=1}^L \sum_{n \in \mathbb{Z}} c_i[n] \varphi_i(t - nT) \right\|^2 \leq B \sum_{i=1}^L \sum_{n \in \mathbb{Z}} |c_i[n]|^2, \quad (3)$$

$$\langle \varphi_i(t - kT), \varphi_j(t - nT) \rangle = \begin{cases} 1 & i = j, k = n \\ 0 & \text{else} \end{cases}. \quad (4)$$

At the same time, the choice of the collection  $\{s_i(t - nT)\}_{k=1,2,\dots,L, n \in \mathbb{Z}}$  forms a frame for  $W$ , In other words, there must exist two constants  $A$  and  $B$  ( $0 < A \leq B < \infty$ ) for every  $x(t) \in W$ , such that:

$$A \|x(t)\|^2 \leq \sum_{i=1}^L \sum_{n \in \mathbb{Z}} |\langle x(t), s_i(t - nT) \rangle|^2 \leq B \|x(t)\|^2. \quad (5)$$

A frame that ceases to be a frame when any one of its element is removed is said to be an exact frame. An exact frame is a Riesz basis. We define the sampling space:

$$S = \left\{ \sum_{i=1}^L \sum_{n \in \mathbb{Z}} c_i[n] \tilde{s}_i(t - nT) : c_i[n] \in l^2 \right\}. \quad (6)$$

In the Hilbert space, knowing  $\langle x(t), s_i(t - nT) \rangle_{i=1,2,\dots,L, n \in \mathbb{Z}}$  is equivalent to knowing the projection of  $Sx$  onto the subspace  $S$ . We want to reconstruct  $x(t)$  from the measurement  $d(n) = (d_1(n) \ d_2(n) \ \dots \ d_L(n))^T$  using a given set of reconstruction vectors  $\{\varphi_i(t - nT)\}_{i=1,2,\dots,L, n \in \mathbb{Z}}$  that span a subspace  $W$ , let us illustrate the generalized sampling process in Fig. 2. For designing the reconstruction operator  $T$ , we start with the following natural requirements:

$$\begin{aligned} D_i(\omega) &= \sum_{n=-\infty}^{+\infty} \sum_{k=1}^L C_k(\omega + 2\pi n) \psi_k(\omega + 2\pi n) S_i(\omega + 2\pi n) \\ &= \sum_{k=1}^L C_k(\omega) \sum_{n=-\infty}^{+\infty} \psi_k(\omega + 2\pi n) S_i(\omega + 2\pi n) \quad i=1, 2, \dots, L, \end{aligned} \quad (7)$$

where  $C_k(\omega)$  and  $D_i(\omega)$  are the discrete-time Fourier transform of  $c_k(n)$  and  $d_i(n)$ .  $\psi_k(\omega)$  and  $S_i(\omega)$  are the Fourier transform of  $\varphi_k(t)$  and  $s_i(-t)$ , respectively.

This leads to a compact relation between the sampling data:

$$d(n) = (d_1(n) \quad d_2(n) \quad \cdots \quad d_L(n))^T$$

and coefficients:

$$c(n) = (c_1(n) \quad c_2(n) \quad \cdots \quad c_L(n))^T$$

via a matrix-vector multiplication in the Fourier domain:

$$H(\omega)C(\omega) = D(\omega), \tag{8}$$

where:

$$- \quad H(\omega) = \begin{pmatrix} h_{1,1}(\omega) & h_{1,2}(\omega) & \cdots & h_{1,L}(\omega) \\ h_{2,1}(\omega) & h_{2,2}(\omega) & \cdots & h_{2,L}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ h_{L,1}(\omega) & h_{L,2}(\omega) & \cdots & h_{L,L}(\omega) \end{pmatrix};$$

$$- \quad h_{i,k}(\omega) = \sum_{n=-\infty}^{+\infty} S_i(\omega + 2\pi n)\psi_k(\omega + 2\pi n);$$

$$- \quad D(\omega) = (D_1(\omega) \quad D_2(\omega) \quad \cdots \quad D_L(\omega))^T;$$

$$- \quad C(\omega) = (C_1(\omega) \quad C_2(\omega) \quad \cdots \quad C_L(\omega))^T.$$

Equation (8) denotes the model of generalized sampling. The traditional method of reconstruction is to solve equation (8).

Unfortunately, the analysis basis functions  $\{s_i(-t)\}_{i=1,2,\dots,L;n \in \mathbb{Z}}$  are non-ideal. Defining the non-ideal basis functions  $\{\hat{s}_i(-t)\}_{i=1,2,\dots,L;n \in \mathbb{Z}}$  by:

$$\hat{S}_i(\omega) = S_i(\omega)\xi_i(\omega) \quad i = 1, 2, \dots, L. \tag{9}$$

From (8), we may obtain  $\hat{C}(\omega)$  from the traditional method.

$$\hat{C}(\omega) = H(\omega)^{-1}\xi(\omega)^{-1}H(\omega)C(\omega), \tag{10}$$

where:

$$\xi(\omega) = \begin{pmatrix} \xi_1(\omega) & 0 & \cdots & 0 \\ 0 & \xi_2(\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_L(\omega) \end{pmatrix}.$$

Analyzing (10), the traditional algorithm will introduce an error  $\xi(\omega)$ . In order to reduce the error  $\xi(\omega)$ , we propose a reconstruction algorithm from least squares criterion.

$$e = \frac{1}{2\pi} \int_0^{2\pi} |C(\omega) - \hat{C}(\omega)|^2 d\omega. \tag{11}$$

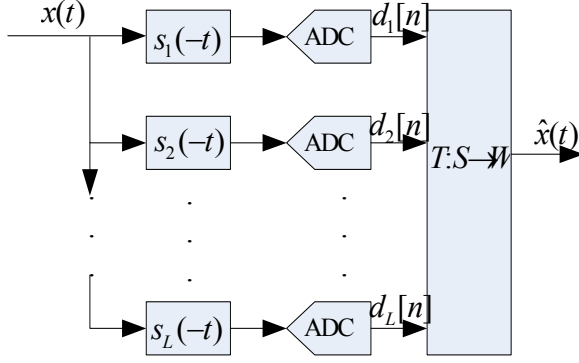


Fig. 1. Generalized sampling.

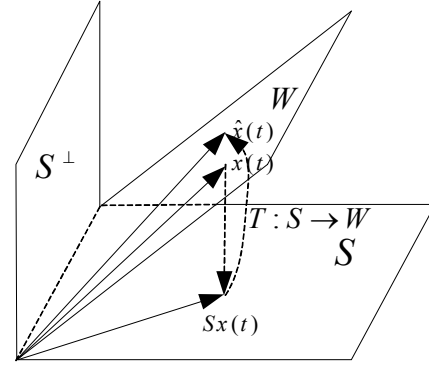


Fig. 2. Geometry illustration of generalized sampling.

### 3. Reconstruction method

In order to obtain  $C(\omega)$ , we add one channel into the generalized sampling system. The corresponding system is schematically represented in Fig. 3.

However, the single channel sampling cannot get an exact  $C(\omega)$ , but it may confirm a distributing range  $\hat{l}$  of  $C(\omega)$ . In order to calibrate  $\hat{C}(\omega)$ , let us project  $\hat{C}(\omega)$  onto  $\hat{l}$ . The reconstruction process is shown in Fig.4. We first reconstruct  $\hat{x}(t)$  from the measurement  $d(n) = (d_1(n) \ d_2(n) \ \dots \ d_L(n))^T$  making use of an oblique projection onto a given reconstruction space, then, confirm a distributing range  $\hat{l}$  of  $C(\omega)$  from  $\{d_0(n)\}_{n \in \mathbb{Z}}$ . Finally, the calibrating algorithm is achieved by projecting  $\hat{C}(\omega)$  onto  $\hat{l}$ .

As illustrated in Fig. 4, the reconstruction method involves the factors  $\hat{l}$  and  $P_i$ , which are abstract operators from the function space to the vector space  $W$ , respectively, and which are not suitable for computer calculations.

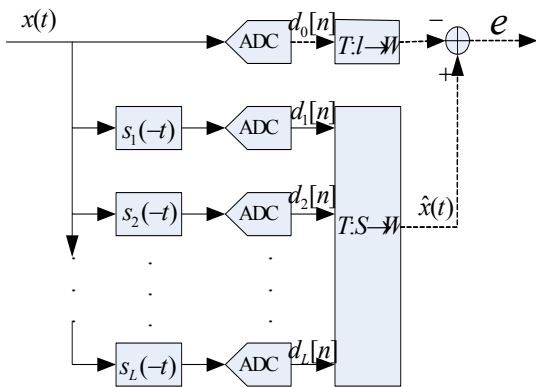


Fig. 3. The improved generalized sampling system.

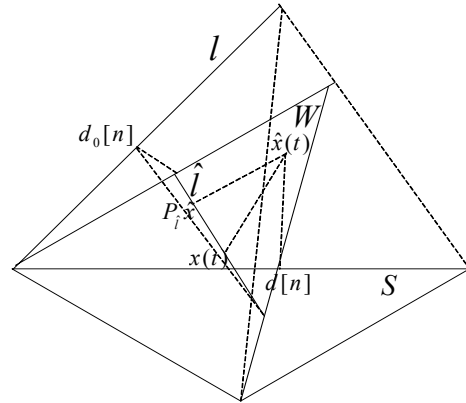


Fig. 4. Geometry illustration of reconstruction method.

In order to specify the algorithm numerically we now provide its matrix expression. For obtaining the distributing range  $\hat{l}$  of  $C(\omega)$ , we start with the following natural requirements:

$$\begin{aligned}
 D_0(\omega) &= \sum_{n=-\infty}^{+\infty} \sum_{i=1}^L C_i(\omega + 2\pi n) \psi_i(\omega + 2\pi n) \\
 &= \sum_{i=1}^L C_i(\omega) \sum_{n=-\infty}^{+\infty} \psi_i(\omega + 2\pi n),
 \end{aligned} \tag{12}$$

where  $D_0(\omega)$  is the discrete-time Fourier transform of  $d_0(n)$ .

Then we have:

$$\Phi(\omega)C(\omega) = D_0(\omega), \tag{13}$$

where:  $\Phi(\omega) = \left( \sum_{n=-\infty}^{+\infty} \psi_1(\omega + 2\pi n), \sum_{n=-\infty}^{+\infty} \psi_2(\omega + 2\pi n), \dots, \sum_{n=-\infty}^{+\infty} \psi_L(\omega + 2\pi n) \right)$ .

Solving the equation (13), the distributing range  $\hat{l}$  of  $C(\omega)$  is derived by:

$$\hat{l}(\omega) = \Phi^-(\omega)D_0(\omega) + (I - \Phi^-(\omega)\Phi(\omega))Z(\omega), \tag{14}$$

where,  $Z(\omega)$  is the discrete-time Fourier transform of  $(z_1[n], z_2[n], \dots, z_L[n])^T$ , and  $(z_1[n], z_2[n], \dots, z_L[n])^T$  is any real numeral.

$$\Phi^-(\omega) = \Phi^H(\omega)(\Phi^H(\omega)\Phi(\omega))^{-1} \tag{15}$$

As shown in Fig. 4, the exact  $C(\omega)$  is not perfectly reproduced in system, but we can find the solution to minimize the quantity:

$$\frac{1}{2\pi} \int_0^{2\pi} |C(\omega) - \hat{C}(\omega)|^2 d\omega. \tag{16}$$

The generalized-inverse is a general way to find the solution to minimize the quantity:

$$\|\hat{C}(\omega) - \Phi^-(\omega)D_0(\omega) - (I - \Phi^-(\omega)\Phi(\omega))Z(\omega)\|^2. \tag{17}$$

As generalized-inverse's properties, the solution is derived by:

$$Z_0(\omega) = G^H(\omega)(F^H(\omega)(I - \Phi^-(\omega)\Phi(\omega))G^H(\omega))^{-1}F^H(\omega)(\hat{C}(\omega) - \Phi^-(\omega)D_0(\omega)) \tag{18}$$

where  $F(\omega)$  is full Col rank,  $G(\omega)$  is full Row rank, and  $F(\omega)G(\omega) = I - \Phi^-(\omega)\Phi(\omega)$ .

#### 4. Error analysis

We discussed the reconstruction method in previous sections. In this section, we briefly analyze the reconstruction error. The reconstruction error is derived by:

$$\begin{aligned}
 \|C(\omega) - \hat{C}(\omega)\|^2 &= \|C(\omega) - C_0(\omega) + C_0(\omega) - \hat{C}(\omega)\|^2 \\
 &= \|C(\omega) - C_0(\omega)\|^2 + \|C_0(\omega) - \hat{C}(\omega)\|^2 + (C(\omega) - C_0(\omega))^H (C_0(\omega) - \hat{C}(\omega)) + (C_0(\omega) - \hat{C}(\omega))^H (C(\omega) - C_0(\omega)),
 \end{aligned} \tag{19}$$

where:

$$C_0(\omega) = \Phi^-(\omega)D_0(\omega) + (I - \Phi^-(\omega)\Phi(\omega))Z_0(\omega).$$

The third term of  $\|C(\omega) - \hat{C}(\omega)\|^2$  is expressed by:

$$\begin{aligned}
 & (C(\omega) - C_0(\omega))^H (C_0(\omega) - \hat{C}(\omega)) \\
 &= (A(\omega)(Z(\omega) - Z_0(\omega)))^H (C_0(\omega) - \hat{C}(\omega)) \\
 &= (A(\omega)(Z(\omega) - Z_0(\omega)))^H (B(\omega) + A(\omega)G^H(C)(F^H(\omega)F(\omega)G(\omega)G^H(\omega))^{-1}F^H(\omega)(\hat{C}(\omega) - B(\omega)) - \hat{C}(\omega)) \quad (20) \\
 &= (Z(\omega) - Z_0(\omega))^H G^H(\omega)F^H(\omega)(F(\omega)G(\omega)G^H(\omega)(F^H(\omega)F(\omega)G(\omega)G^H(\omega))^{-1}F^H(\omega) - I)\hat{C}(\omega) \\
 &\quad - (Z(\omega) - Z_0(\omega))^H G^H(\omega)F^H(\omega)(F(\omega)G(\omega)G^H(\omega)(F^H(\omega)F(\omega)G(\omega)G^H(\omega))^{-1}F^H(\omega) - I)B(\omega) \\
 &= 0,
 \end{aligned}$$

where:

$$A(\omega) = I - \Phi^-(\omega)\Phi(\omega), \quad B(\omega) = \Phi^-(\omega)D_0(\omega).$$

Then we have:

$$\|C(\omega) - \hat{C}(\omega)\|^2 = \|C(\omega) - C_0(\omega)\|^2 + \|C_0(\omega) - \hat{C}(\omega)\|^2 \geq \|C(\omega) - C_0(\omega)\|^2. \quad (21)$$

Further, (21) implies that the reconstruction error is less the proposed algorithm than the traditional reconstruction algorithm. From Fig. 4,  $\|C(\omega) - C_0(\omega)\|^2$  is also expressed by:

$$\begin{aligned}
 \|P_i(\hat{x}(t) - x(t))\|^2 &= \frac{\langle \hat{x}(t) - x(t), P_i(\hat{x}(t) - x(t)) \rangle^2}{\|\hat{x}(t) - x(t)\|^2 \|P_i(\hat{x}(t) - x(t))\|^2} \|\hat{x}(t) - x(t)\|^2 \\
 &= \cos^2(\alpha(\hat{x}(t) - x(t), \hat{l})) \|\hat{x}(t) - x(t)\|^2,
 \end{aligned} \quad (22)$$

where:

$$\alpha(\hat{x}(t) - x(t), \hat{l}) = \min_{y(t) \in \hat{l}} \alpha(\hat{x}(t) - x(t), y(t)). \quad (23)$$

## 5. Experiment and analysis

We now apply the proposed approach to reconstruction from samples of Oscilloscope. The Oscilloscope is schematically represented in Fig. 3. In the Oscilloscope, generalized sampling is selected. So we utilize the proposed approach to recover generalized samples and compare with the existing technique.

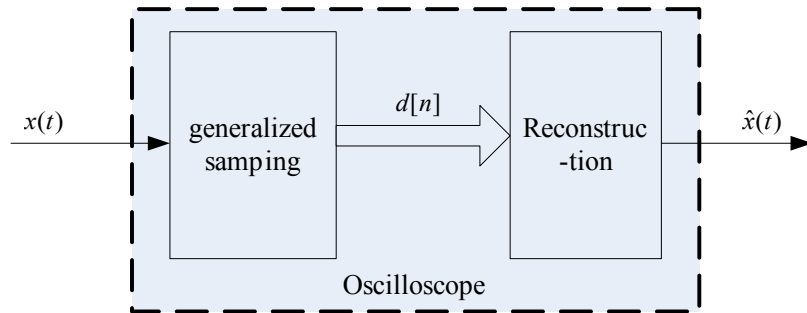


Fig. 4. The structure of the digitizer.

In this section, we consider the case  $L=2$ . The corresponding analysis filters in the block diagram in Fig. 1 are  $s_1(-t) = \delta(t - T/4)$  and  $s_2(-t) = \delta(t - T/3)$ . The generating functions  $\varphi_1(t)$  and  $\varphi_2(t)$  of reconstruction space  $W$  are given by:

$$\varphi_1(t) = \sin c\left(\frac{t}{T}\right) \exp(j2\pi \frac{5}{6T}t),$$

$$\varphi_2(t) = \sin c\left(\frac{t}{T}\right) \exp\left(-j2\pi \frac{5}{6T}t\right),$$

where  $T$  is the sampling period.

In order to prove the proposed method, we assume  $\xi(\omega)$  that:

$$\xi(\omega) = \begin{pmatrix} e^{-j\omega 0.005} & 0 \\ 0 & e^{j\omega\delta} \end{pmatrix}.$$

To investigate the validity of the proposed method, we computed the  $\cos(\alpha)$  number (23) as a function of the parameter  $\delta$ . The graph in Fig. 5 shows that  $\cos(\alpha)$  has the least value round  $\delta = 0.005$  for  $0.001 \leq \delta \leq 0.009$ . To compare the proposed method and the traditional algorithm, we have included the graph for the SNR of two methods in Fig. 6, the SNR of the proposed method and traditional algorithm are marked by ‘\*’ and ‘•’. The proposed method has the most favorable behavior around  $\delta = 0.005$  for  $0.001 \leq \delta \leq 0.009$  from Fig. 6.

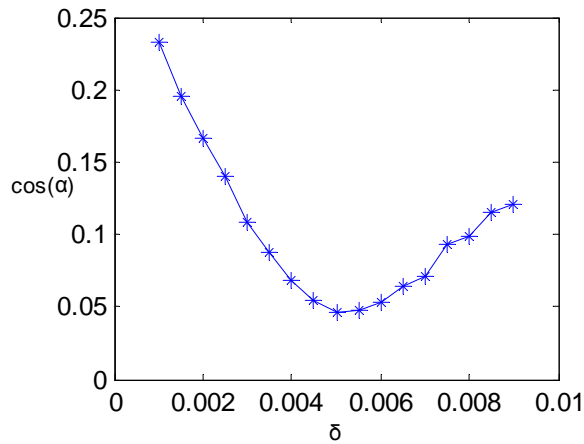


Fig. 5.  $\cos(\alpha)$  as a function of the parameter  $\delta$ .

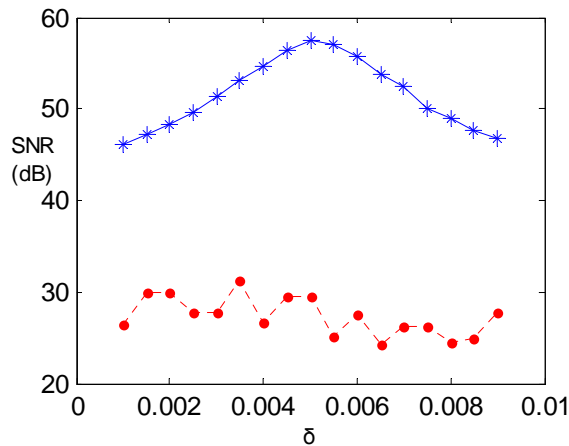


Fig. 6. SNR as a function of the parameter  $\delta$ .

To further prove the validity of the proposed method, we now present an application of the proposed method to a signal reconstruction problem. Assuming the input signal:

$$x(t) = 2 \cos\left(\frac{5}{6} \pi \times 10^8 t\right) \sin\left(\frac{1}{3} \pi \times 10^8 t\right)$$

and sampling period  $T = 1/10^8$ . Fig. 7a and Fig. 7b show the reconstruction error by the proposed method and the traditional algorithm. In order to facilitate the comparison, the original spectrum is indicated by Fig. 8a. Fig. 8b and Fig. 8c show the spectrum from reconstruction continuous signal by the proposed method and the traditional algorithm, respectively. We can see blocking artifacts in Fig. 8b, while there is no such effect in Fig. 9c.

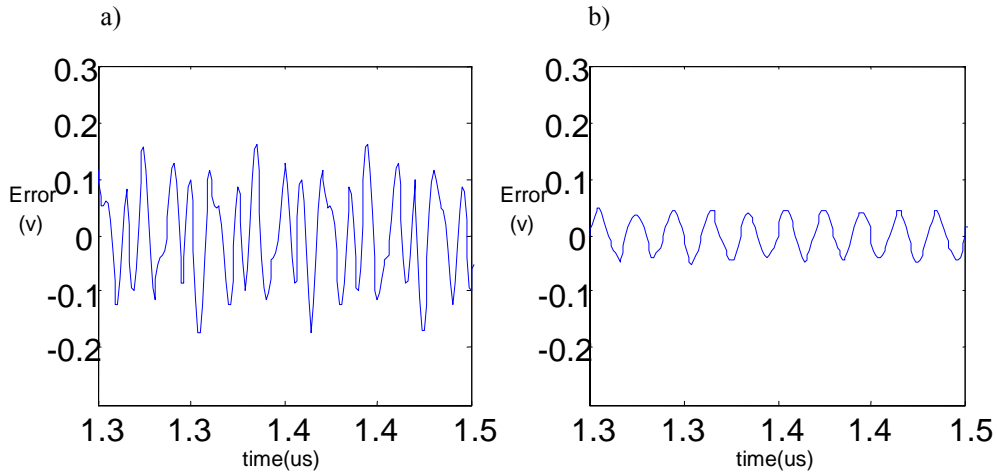


Fig. 7. Reconstruction error. a) The traditional algorithm. b) The proposed method.

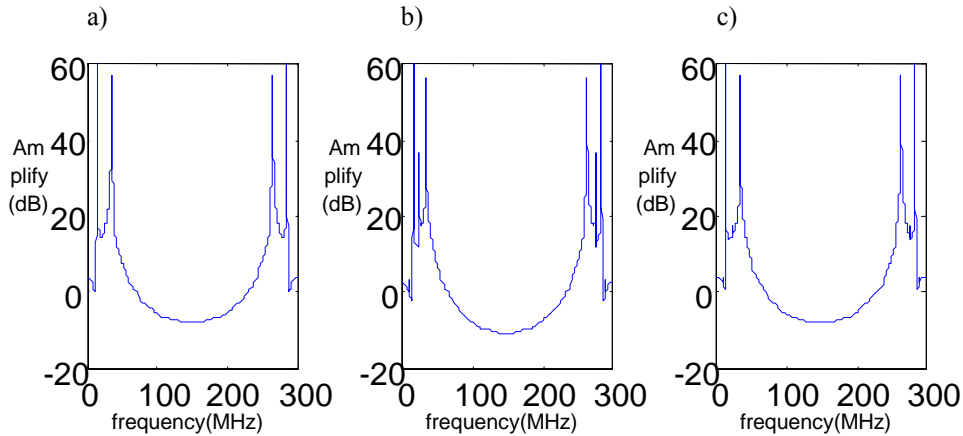


Fig. 8. Spectrum. a) Original. b) The traditional algorithm. c) The proposed method.

## 6. Conclusion

We proposed the reconstruction method of generalized sampling that yields a well-defined function even in the case where the analysis basis functions are non-ideal. The method is to add one channel into the generalized sampling system. And the reconstruction algorithm makes use of the generalized inverse and has a simple geometric interpretation.

Finally, we compared the proposed method with the traditional algorithm, and demonstrated both analytically and through simulation that the proposed method can often outperform the generalized sampling reconstruction strategy.



## References

- [1] A. Papoulis: “Generalized sampling expansion”. *IEEE Trans. Circuits Syst.*, vol. 24, 1977, pp. 652–654.
- [2] M. Unser, A. Aldroubi: “A general sampling theory for nonideal acquisition devices”. *IEEE Trans. Signal Process.*, vol. 42, 1994, pp. 2915–2925.
- [3] S. Chang Eon, L. Mun Bae, R. Kyung Soo: “Nonuniform Sampling of Bandlimited Functions”. *IEEE Trans. Informat. Theory*, vol. 54, no. 7, 2008, pp. 3814–3819.
- [4] M. Unser: “Splines: A perfect fit for signal and image processing”. *IEEE Signal Process. Mag.*, 1999, pp. 22–38.
- [5] Y.C. Eldar, M. Unser: “Non-ideal sampling and interpolation from noisy observations in shift-invariant spaces”. *IEEE Trans. Signal Process.*, vol. 54, no. 7, 2006, pp. 2636–2651.
- [6] S. Remani, M. Unser: “Nonideal Sampling and Regularization Theory”. *IEEE Trans. Signal Process.*, vol. 56, no. 3, 2008, pp. 1055–1070.
- [7] Y.C. Eldar: “Sampling and reconstruction in arbitrary spaces and oblique dual frame vectors”. *J. Fourier Anal. Appl.*, vol. 1, no. 9, Jan. 2003, pp. 77–96.
- [8] P.P. Vaidyanathan: “Generalizations of the sampling theorem: Seven decades after Nyquist”. *IEEE Trans. Circuit Syst. I. Fundam. Theory Appl.*, vol. 48, no. 9, 2001, pp. 1094–1109.
- [9] M. Lu Yue, N. Do Minh: “A Theory for Sampling Signals From a Union of Subspaces”. *IEEE Trans. Signal Process.*, vol. 56, no. 6, 2008, pp. 2334–2345.
- [10] I. Djokovic, P.P. Vaidyanathan: “Generalized sampling theorems in multiresolution subspaces”. *IEEE Trans. Signal Process.*, vol. 45, Mar. 1997, pp. 583–599.
- [11] Z. Zhaoxuan, W. Houjun, W. Zhigang: “Computation of reconstruction function for samples in shift-invariant spaces”. *Metrol. Meas. Syst.*, vol. XVI, no. 4, 2009, pp. 535–544.
- [12] Y.C. Eldar: “Sampling without input constraints: Consistent reconstruction in arbitrary spaces”. *Sampling, Wavelets and Tomography*, A.I. Zayed, J.J. Benedetto, Eds. Boston, MA: Birkhauser, 2004, pp. 3360.